A weak Fano quadric surface bundle with a bisectional flopping curve

Kiyohiko TAKEUCHI

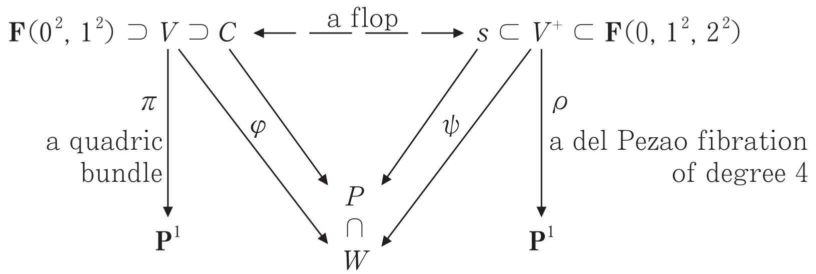
Abstract

This note gives an interesting example for smooth weak Fano 3-fold V with quadric bundle structure which has only one flopping curve. The weak Fano 3-fold V has the flop V^+ with del Pezzo fibration of degree 4. This is a counterexample of Vologodsky's result[1] [5] that says two del Pezzo fibrations differing only by a flop are of the same degree. The example has been constructed in [4].

1. Introduction

A smooth projective 3-fold V is a *weak Fano 3-fold* if its anti-canonical divisor $-K_V$ is nef and big (cf. [3]). Here a divisor $D \subset V$ is *nef* if $(D,C) \geq 0$ for any effective curve $C \subset V$, and a nef divisor D is *big* if $(D^3) > 0$. For a vector bundle $\mathcal{E}$ of rank r over the projective line $\mathbf{P}^1$, we can construct the projective bundle $\mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$. Since any vector bundle over $\mathbf{P}^1$ can be decomposed into a direct sum of line bundles, for sake of simplicity, $\mathbf{F}(a_0, a_1, \ldots, a_n)$ denotes $\mathbf{P}(\mathcal{E})$ for $\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}(a_i)$ over $\mathbf{P}^1$. For example, $\mathbf{F}(0, a)$ is the Hirzebruch surface with negative section s, $s^2 = -a$, and $\mathbf{F}(0^3)$ is the direct product $\mathbf{P}^2 \times \mathbf{P}^1$.

We here construct a smooth weak Fano 3-fold V with quadric bundle structure which has a flop V^+ with del Pezzo fibration of degree 4:



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[1] Vologodsky calculated the Euler number of the rational surface containing flopped curve in [5]. Although it is possible for the surface to have singularities along the flopped curve, he treated the surface as a smooth one. This is the reason why he overlooked our case.

2. Construction of the 3-fold V

Consider the vector bundle $\mathcal{E}=\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(1)^{\oplus 2}$ of rank 4 over the projective line $\mathbf{P}^1$, and the $\mathbf{P}^3$-bundle $\pi: X=\mathbf{P}(\mathcal{E})=\mathbf{F}(0^2, 1^2)\to\mathbf{P}^1$. Let H and F be the tautological divisor and a fiber of $\pi: \mathrm{X}\to\mathbf{P}^1$, respectively.

Let $V\subset X$ be a smooth 3-fold linearly equivalent to $2H+F$ as a divisor on X. The 3-fold V is a quadric surface bundle with Picard number 2, because $V\sim 2H+F$ is an ample divisor of the $\mathbf{P}^3$-bundle $X=\mathbf{F}(0^2, 1^2)$ over $\mathbf{P}^1$. Therefore, Pic V is generated by the restrictions H_V and F_V of H and F to V, i.e., Pic $V=\mathbf{Z}[H_V]\oplus\mathbf{Z}[F_V]$, and hence $N^1(V)=\mathbf{R}[H_V]\oplus\mathbf{R}[F_V]\cong N^1(X)=\mathbf{R}[H]\oplus\mathbf{R}[F]$. By the duality, $N_1(V)\cong N_1(X)=\mathbf{R}[l]\oplus\mathbf{R}[s]$, where l and s are the dual basis with respect to H and F and they are a line in a fiber $F\cong\mathbf{P}^3$ and the minimal section of $\pi: X\to\mathbf{P}^1$, respectively. Any effective curve C in V is considered as in X, and numerically equivalent to $al+bs$ for some non-negative integers $a, b\in\mathbf{Z}_{\geq 0}$.

The adjunction formula gives $-K_V=(-K_V-V)|_V=2H_V-F_V$. For an effective curve $C\equiv al+bs$ in V , the intersection number $(C\cdot -K_V)\leq 0$ implies $(0\leq)2a\leq b$. The curve C is mapped onto $\mathbf{P}^1$ by the projection $\pi: X\to\mathbf{P}^1$. Let $\nu: D\to C\subset X$ be the normalization of C, and $\mu=\pi\circ\nu: D\to\mathbf{P}^1$. We have $\deg\mu=b>0$ and $\nu^*\mathcal{O}_X(1)=\mathcal{O}_D(a)$. From $\pi^*\mathcal{E}\to\mathcal{O}_X(1)\to 0$ and $\mu^*\mathcal{O}(1)=\mathcal{O}_D(b)$, it follows that

$$\pi^*\mathcal{E}=\mathcal{O}_D^{\oplus 2}\oplus\mathcal{O}_D(b)^{\oplus 2}\to\mathcal{O}_D(a)\to 0.$$

Since $a<b$, this sequence factors $\mathcal{O}_D^{\oplus 2}\to\mathcal{O}_D(a)$, hence the morphism $\nu: D\to X$ factors $D\to\mathbf{P}(\mathcal{O}^{\oplus 2})=\mathbf{F}(0^2)\subset\mathrm{X}$, i.e., the curve C is on the ruled surface $\mathbf{F}(0^2)$. It follows from $V\sim 2H+F$ that the intersection curve $V\cap\mathbf{F}(0^2)$ is numerically equivalent to $l+2s$, which is no other than the curve C. Thus V has only one curve $C\equiv l+2s$ with $(-K_V\cdot C)=0$, which is a flopping curve (cf. [1], [2]). Moreover $-K_V$ is nef and big because

$$(-K_V)^3=(2H_V-F_V)^3=(2H-F)^3(2H+F)=16(H^4-H^3F)=16>0.$$

Consequently, the smooth 3-fold $V\sim 2H+F$ in $X=\mathbf{F}(0^2, 1^2)$ is a smooth weak Fano 3-fold with Picard number 2, which has only one flopping curve $C\equiv l+2s$ as a bisection of the quadric bundle structure $\pi: V\to\mathbf{P}^1$.

3. Construction of the flop V^+

In [4], we has already constructed the flop V^+ of this quadric bundle Fano 3-fold V. We here reconstruct more concretely by using bihomogeneous coordinates system of $\mathbf{P}^3$-bundle $X = \mathbf{F}(0^2, 1^2)$.

The bihomogeneous coordinates ring of X is $R_X = \mathbf{C}[x_0, x_1, y_2, y_3, s_0, s_1]$ with bidegree deg $x_i = (1, 0)$, deg $y_i = (1, -1)$ and deg $s_j = (0, 1)$. The projective space bundle X has a standard affine chart $X = \cup_{i=0,1,2,3\,;\,j=0,1} U_{ij}$. After making linear transformations if it is necessary, the 3-fold V is defined by the bihomogeneous polynomial

$$
\begin{aligned}
&f(x_0, x_1, y_2, y_3, s_0, s_1)\\
&\quad = x_0 x_1 s_0 + (x_0^2 + x_1^2) s_1 + x_1\big(l_0(y) s_0^2 + l_1(y) s_0 s_1 + l_2(y) s_1^2\big)\\
&\qquad\qquad + q_0(y) s_0^3 + q_1(y) s_0^2 s_1 + q_2(y) s_0 s_1^2 + q_3(y) s_1^3\\
&\quad = f_0(x_0, x_1, y_2, y_3, s_0) s_0 + f_1(x_0, x_1, y_2, y_3, s_0, s_1) s_1
\end{aligned}
$$

of $\deg f = (2, 1)$ for the linear forms $l_i(y) = l_{i0} y_2 + l_{i1} y_3$ and the quadric forms $q_i(y) = q_{i0} y_2^2 + q_{i1} y_2 y_3 + q_{i2} y_3^2$, where

$$
\begin{aligned}
&f_0 = x_0 x_1 + x_1 l_0(y) s_0 + q_0(y) s_0^2, \qquad \text{and}\\
&f_1 = x_0^2 + x_1^2 + x_1\big(l_1(y) s_0 + l_2(y) s_1\big) + \big(q_1(y) s_0^2 + q_2(y) s_0 s_1 + q_3(y) s_1^2\big)
\end{aligned}
$$

Let R_V be the quotient ring $R_X/(f)$.

Consider the homogeneous coordinates ring

$$R_Z = \mathbf{C}\big[p, a_0, a_1, b_0, b_1, c_0, c_1, c_2, d_0, d_1, d_2\big]$$

of $\mathbf{P}^{10}$. The ring homomorphisms $\varphi_j^\sharp : R_Z \to R_V\left[\frac{1}{s_j}\right]$ $(j = 0, 1)$ defined by

$$
\varphi_0^\sharp : \quad
\begin{array}{ll}
p \mapsto f_1(x_0, x_1, y_2, y_3, s_0, s_1)/s_0 & c_0 \mapsto y_2^2 s_0\\
a_0 \mapsto x_0 y_2 & c_1 \mapsto y_2 y_3 s_0\\
a_1 \mapsto x_0 y_3 & c_2 \mapsto y_3^2 s_0\\
b_0 \mapsto x_1 y_2 & d_0 \mapsto y_2^2 s_1\\
b_1 \mapsto x_1 y_3 & d_1 \mapsto y_2 y_3 s_1\\
 & d_2 \mapsto y_3^2 s_1
\end{array}
$$

$$\varphi_1^\sharp: \quad \begin{aligned} p &\mapsto f_0(x_0, x_1, y_2, y_3, s_0)/s_1 \\ a_0 &\mapsto x_0 y_2 \\ a_1 &\mapsto x_0 y_3 \\ b_0 &\mapsto x_1 y_2 \\ b_1 &\mapsto x_1 y_3 \end{aligned} \qquad \begin{aligned} c_0 &\mapsto y_2{}^2 s_0 \\ c_1 &\mapsto y_2 y_3 s_0 \\ c_2 &\mapsto y_3{}^2 s_0 \\ d_0 &\mapsto y_2{}^2 s_1 \\ d_1 &\mapsto y_2 y_3 s_1 \\ d_2 &\mapsto y_3{}^2 s_1 \end{aligned}$$

can be glued because

$$f_1\big(x_0, x_1, y_2, y_3, s_0, s_1\big)/s_0 = f_0\big(x_0, x_1, y_2, y_3, s_0\big)/s_1 \quad \text{in} \quad R_V\Big[\frac{1}{s_0}, \frac{1}{s_1}\Big].$$

Let $\varphi : V \to Z$ be the morphism determined by the ring homomorphisms $\varphi_i^\sharp$. The kernels of $\varphi_0^\sharp$ and $\varphi_1^\sharp$ coincide, and equal to the homogeneous ideal

$$\begin{aligned} I = \Big(& d_0 p + f_0^{(0)}, d_1 p + f_0^{(1)}, d_2 p + f_0^{(2)}, c_0 p - f_1^{(1)}, c_1 p - f_1^{(1)}, c_2 p - f_1^{(2)}, \\ & [ab], [ac_0], [ac_1], [ad_0], [ad_1], \\ & [bc_0], [bc_1], [bd_0], [bd_1], [c_0 c_1], [c_0 d_0], [c_0 d_1], [c_1 d_0], [c_1 d_1], [d_0 d_1]\Big). \end{aligned}$$

Here the polynomials $f_0^{(i)}$ and $f_1^{(i)}$ are

$$\begin{aligned} f_0^{(0)} &= a_0 b_0 + b_0\big(l_{00} c_0 + l_{01} c_1\big) + \big(q_{00} c_0{}^2 + q_{01} c_0 c_1 + q_{02} c_1{}^2\big), \\ f_0^{(1)} &= a_0 b_1 + b_1\big(l_{00} c_0 + l_{01} c_1\big) + \big(q_{00} c_0 c_1 + q_{01} c_1{}^2 + q_{02} c_1 c_2\big), \\ f_0^{(2)} &= a_1 b_1 + b_1\big(l_{00} c_1 + l_{01} c_2\big) + \big(q_{00} c_1{}^2 + q_{01} c_1 c_2 + q_{02} c_2{}^2\big), \\ f_1^{(0)} &= a_0{}^2 + b_0{}^2 + b_0\Big(\big(l_{10} c_0 + l_{11} c_1\big) + \big(l_{20} d_0 + l_{21} d_1\big)\Big) \\ &\quad + \big(q_{10} c_0{}^2 + q_{11} c_0 c_1 + q_{12} c_1{}^2\big) + \big(q_{20} c_0 d_0 + q_{21} c_0 d_1 + q_{22} c_1 d_1\big) \\ &\quad + \big(q_{30} d_0{}^2 + q_{31} d_0 d_1 + q_{32} d_1{}^2\big), \\ f_1^{(1)} &= a_0 a_1 + b_0 b_1 + b_1\Big(\big(l_{10} c_0 + l_{11} c_1\big) + \big(l_{20} d_0 + l_{21} d_1\big)\Big) \\ &\quad + \big(q_{10} c_0 c_1 + q_{11} c_1{}^2 + q_{12} c_1 c_2\big) + \big(q_{20} c_0 d_1 + q_{21} c_1 d_1 + q_{22} c_1 d_2\big) \\ &\quad + \big(q_{30} d_0 d_1 + q_{31} d_1 d_1 + q_{32} d_1 d_2\big), \\ f_1^{(2)} &= a_1{}^2 + b_1{}^2 + b_1\Big(\big(l_{10} c_1 + l_{11} c_2\big) + \big(l_{20} d_1 + l_{21} d_2\big)\Big) \\ &\quad + \big(q_{10} c_1{}^2 + q_{11} c_1 c_2 + q_{12} c_2{}^2\big) + \big(q_{20} c_1 d_1 + q_{21} c_1 d_2 + q_{22} c_2 d_2\big) \\ &\quad + \big(q_{30} d_1{}^2 + q_{31} d_1 d_2 + q_{32} d_2{}^2\big), \end{aligned}$$

and the symbol [ ] is an abbreviation of $[ab]=a_0b_1-a_1b_0$, $[ac_0]=a_0c_1-a_1c_0$, $[ac_1]=a_0c_2-a_1c_1$ and so on. The ideal I determines the subvariety $W\subset Z$. Denote by R_W the quotient ring R_Z/I.

We will calculate the image $\varphi(V_{ij})$ for each affine open set $V_{ij}=V\cap U_{ij}$($i=0, 1, 2, 3$; $j=0, 1$) of V. The image of

$$V_{20}=\mathrm{Spec}\ \mathbf{C}\left[\frac{x_0}{y_2s_0}, \frac{x_1}{y_2s_0}, \frac{y_3}{y_2}, \frac{s_1}{s_0}\right]\Big/\left(f\left(\frac{x_0}{y_2s_0}, \frac{x_1}{y_2s_0}, 1, \frac{y_3}{y_2}, 1, \frac{s_1}{s_0}\right)\right)$$

coincides with

$$\begin{aligned}
&W\cap Z_{c_0}\\
&\quad=W\cap\mathrm{Spec}\ \mathbf{C}\left[\frac{p}{c_0}, \frac{a_0}{c_0}, \frac{a_1}{c_0}, \frac{b_0}{c_0}, \frac{b_1}{c_0}, \frac{c_1}{c_0}, \frac{c_2}{c_0}, \frac{d_0}{c_0}, \frac{d_1}{c_0}, \frac{d_2}{c_0}\right]\\
&\quad=\mathrm{Spec}\ \mathbf{C}\left[\frac{a_0}{c_0}, \frac{b_0}{c_0}, \frac{c_1}{c_0}, \frac{d_0}{c_0}\right]\Big/\left(f_0\left(\frac{a_0}{c_0}, \frac{b_0}{c_0}, 1, \frac{c_1}{c_0}, 1\right)+\frac{d_0}{c_0}f_1\left(\frac{a_0}{c_0}, \frac{b_0}{c_0}, 1, \frac{c_1}{c_0}, 1\frac{d_0}{c_0}\right)\right)\\
&\quad=\mathrm{Spec}\ \mathbf{C}\left[\frac{a_0}{c_0}, \frac{b_0}{c_0}, \frac{c_1}{c_0}, \frac{d_0}{c_0}\right]\Big/\left(f\left(\frac{a_0}{c_0}, \frac{b_0}{c_0}, 1, \frac{c_1}{c_0}, 1, \frac{d_0}{c_0}\right)\right),
\end{aligned}$$

and the restriction map $\varphi|_{V_{20}}: V_{20}\to W_{c_0}:=W\cap Z_{c_0}$ is an isomorphism. Similarly, three restriction maps $\varphi|_{V_{21}}: V_{21}\to W_{d_0}$, $\varphi|_{V_{30}}: V_{30}\to W_{c_2}$, and $\varphi|_{V_{31}}: V_{31}\to W_{d_2}$ are isomorphisms. The image $W_0=\varphi(V_{00}\cup V_{01})$ of the union of

$$\begin{cases}
V_{00}=\mathrm{Spec}\ \mathbf{C}\left[\frac{x_1}{x_0}, \frac{y_2s_0}{x_0}, \frac{y_3s_0}{x_0}, \frac{s_1}{s_0}\right]\Big/\left(f\left(1, \frac{x_1}{x_0}, \frac{y_2s_0}{x_0}, \frac{y_3s_0}{x_0}, 1, \frac{s_1}{s_0}\right)\right) \qquad \text{and}\\
V_{01}=\mathrm{Spec}\ \mathbf{C}\left[\frac{x_1}{x_0}, \frac{y_2s_1}{x_0}, \frac{y_3s_1}{x_0}, \frac{s_0}{s_1}\right]\Big/\left(f\left(1, \frac{x_1}{x_0}, \frac{y_2s_1}{x_0}, \frac{y_3s_1}{x_0}, \frac{s_0}{s_1}, 1\right)\right)
\end{cases}$$

coincides with the union $W_{a_0}\cup W_{a_1}\cup P$, where

$$\begin{aligned}
W_{a_0}&=W\cap\mathrm{Spec}\ \mathbf{C}\left[\frac{p}{a_0}, \frac{a_1}{a_0}, \frac{b_0}{a_0}, \frac{b_1}{a_0}, \frac{c_0}{a_0}, \frac{c_1}{a_0}, \frac{c_2}{a_0}, \frac{d_0}{a_0}, \frac{d_1}{a_0}, \frac{d_2}{a_0}\right]\\
&=\mathrm{Spec}\ \mathbf{C}\left[\frac{p}{a_0}, \frac{a_1}{a_0}, \frac{b_0}{a_0}, \frac{c_0}{a_0}, \frac{d_0}{a_0}\right]\Bigg/\begin{pmatrix} f_0\left(1, \frac{b_0}{a_0}, 1, \frac{a_1}{a_0}, \frac{c_0}{a_0}\right)+\frac{p}{a_0}\frac{d_0}{a_0}\\ f_1\left(1, \frac{b_0}{a_0}, 1, \frac{a_1}{a_0}, \frac{c_0}{a_0}, \frac{d_0}{c_0}\right)-\frac{p}{a_0}\frac{c_0}{a_0}\end{pmatrix},\\
W_{a_1}&=W\cap\mathrm{Spec}\ \mathbf{C}\left[\frac{p}{a_1}, \frac{a_0}{a_1}, \frac{b_0}{a_1}, \frac{b_1}{a_1}, \frac{c_0}{a_1}, \frac{c_1}{a_1}, \frac{c_2}{a_1}, \frac{d_0}{a_1}, \frac{d_1}{a_1}, \frac{d_2}{a_1}\right]\\
&=\mathrm{Spec}\ \mathbf{C}\left[\frac{p}{a_1}, \frac{a_0}{a_1}, \frac{b_1}{a_1}, \frac{c_2}{a_1}, \frac{d_2}{a_1}\right]\Bigg/\begin{pmatrix} f_0\left(1, \frac{b_1}{a_1}, \frac{a_0}{a_1}, 1, \frac{c_2}{a_1}\right)+\frac{p}{a_1}\frac{d_2}{a_1}\\ f_1\left(1, \frac{b_1}{a_1}, \frac{a_0}{a_1}, 1, \frac{c_2}{a_1}, \frac{d_2}{c_1}\right)-\frac{p}{a_1}\frac{c_2}{a_1}\end{pmatrix},
\end{aligned}$$

and $P=\{[1:0:\cdots:0]\}\in W\subset Z\cong\mathbf{P}^{10}$. Indeed, one of $\frac{c_0}{a_0}$ and $\frac{d_0}{a_0}$ $\left(\text{resp.}\frac{c_2}{a_1}\text{ and }\frac{d_2}{a_1}\right)$ is not 0 on W_{a_0} $\left(\text{resp. }W_{a_1}\right)$, and hence the part $\left(y_2\neq 0\right)$ $\left(\text{resp.}\left(y_3\neq 0\right)\right)$ in $V_{00}\cup V_{01}$ is isomorphic to $W_{a_0}\left(\text{resp.}W_{a_1}\right)$ by φ and the image of $\left(y_2=y_3=0\right)\subset V_{00}\cup V_{01}$ is $P\in W$. Therefore $\varphi: V_{00}\cup V_{01}\to W_{a_0}\cup W_{a_1}\cup P$ is an isomorphism outside P. Similarly, $\varphi: V_{10}\cup V_{11}\to W_{b_0}\cup W_{b_1}\cup P$ is an isomorphism outside P. The inverse image $\varphi^{-1}(P)\subset V$ of $P\in W$ is a curve $C=\left(y_2=y_3=0\right)\subset V$ on $\mathbf{F}(0^2)\cong\mathbf{P}^1\times\mathbf{P}^1$, which is the unique flopping curve on V.

Now let ρ : $Y=\mathbf{F}(0,\ 1^2,\ 2^2)\to\mathbf{P}^1$ be the $\mathbf{P}^4$-bundle over $\mathbf{P}^1$ having bihomogeneous coordinates ring $R_Y=\mathbf{C}\left[u_0, v_1, v_2, w_3, w_4, t_0, t_1\right]$ with bidegree deg $u_0=(1,\ 0)$, deg $v_i=(1,\ -1)$, deg $w_i=(1,\ -2)$ and deg $t_j=(0,\ 1)$. Consider two bihomogeneous polynomials

$$\begin{aligned}g_0&=w_4u_0+f_0\left(v_1, v_2, t_0, t_1, w_3\right)\\&=w_4u_0+\left(v_1v_2+v_2l_0(t)w_3+q_0(t)w_3{}^2\right),\\g_1&=w_3u_0-f_1\left(v_1, v_2, t_0, t_1, w_3, w_4\right)\\&=w_3u_0-\left(v_1^2+v_2^2+v_2\left(l_1(t)w_3+l_2(t)w_4\right)+q_1(t)w_3^2+q_2(t)w_3w_4+q_3(t)w_4^2\right)\end{aligned}$$

of bidegree deg $g_0=$deg $g_1=(2,\ -2)$, and the subvariety $V^+\subset Y=\mathbf{F}(0,\ 1^1,\ 2^2)$ defined by the ideal $I^+=\left(g_0, g_1\right)$ in R_Y. The variety V^+ is an S_4-bundle, del Pezzo fibration of degree 4, over $\mathbf{P}^1$. The ring homomorphism $\psi^\#: R_Z\to R_Y$ defined by

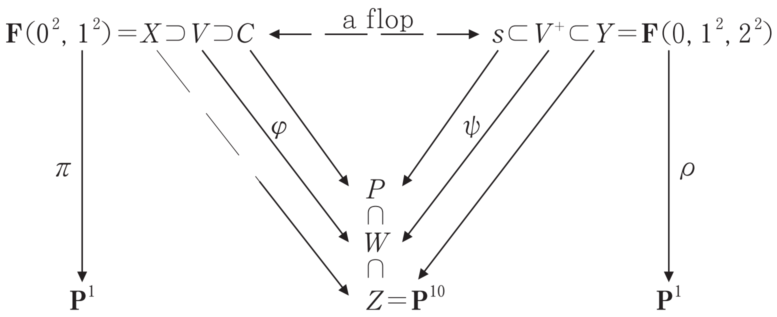
$$\psi^\#: p\mapsto u_0\qquad \begin{matrix}a_0\mapsto v_1t_0\\ a_1\mapsto v_1t_1\end{matrix}\qquad \begin{matrix}b_0\mapsto v_2t_0\\ b_1\mapsto v_2t_1\end{matrix}\qquad \begin{matrix}c_0\mapsto w_3t_0{}^2\\ c_1\mapsto w_3t_0t_1\\ c_2\mapsto w_3t_1{}^2\end{matrix}\qquad \begin{matrix}d_0\mapsto w_4t_0{}^2\\ d_1\mapsto w_4t_0t_1\\ d_2\mapsto w_4t_1{}^2\end{matrix}$$

corresponds to the morphism $\psi: Y\to Z$. This morphism is defined by the linear system $\left|H_Y\right|$ on Y, where H_Y are the tautological line bundle of the $\mathbf{P}^4$-bundle $\rho: Y\to\mathbf{P}^1$. Hence the morphism ψ is an isomorphism outside of the minimal section $s=\mathbf{F}(0)\subset Y\to\mathbf{P}^1$ associated to the surjection $\mathcal{O}\oplus\mathcal{O}(1)^{\oplus 2}\oplus\mathcal{O}(2)^{\oplus 2}\to\mathcal{O}$. The image $\psi(s)$ is the point $P=\{[1:0:\cdots:0]\}\in Z=\mathbf{P}^{10}$. The image $\psi(V^+)\subset Z$ is determined by the ideal

$$\begin{aligned}\psi^{\#-1}I^+=\Big(&d_0p+f_0^{(0)}, d_1p+f_0^{(1)}, d_2p+f_0^{(2)}, c_0p-f_1^{(0)}, c_1p-f_1^{(1)}, c_2p-f_1^{(2)},\\&\left[ab\right], \left[ac_0\right], \left[ac_1\right], \left[ad_0\right], \left[ad_1\right],\\&\left[bc_0\right], \left[bc_1\right], \left[bd_0\right], \left[bd_1\right], \left[c_0c_1\right], \left[c_0d_0\right], \left[c_0d_1\right], \left[c_1d_0\right], \left[c_1d_1\right], \left[d_0d_1\right]\Big).\end{aligned}$$

This is nothing else but the ideal $I \subset R_Z$ defining the variety $W = \varphi(V) \subset Z$. The restricted morphism $\psi : V^+ \to W$ is an isomorphism outside $\psi : s \to P$.

Thus we obtain the following diagram:



and $V \cdots\to V^+$ is the flop corresponding $C \subset V$ to $s \subset V^+$. Here the flopping curve $C \subset V$ is a bisection of the quadric surface bundle $\pi : V \to \mathbf{P}^1$, and the flopped curve $s \subset V^+$ is a section of the del Pezzo fibration $\rho : V^+ \to \mathbf{P}^1$ of degree 4. The map φ is not defined globally on X, but is defined on V itself.

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