Stability analysis of numerical methods for stochastic systems with additive noise

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Abstract Stochastic differential equations (SDEs) represent physical phenomena dominated by stochastic processes. As for deterministic ordinary differential equations (ODEs), various numerical methods are proposed for SDEs. We have proposed two types of numerical stability for SDEs, namely *mean-square* stability and *trajectory* stability. However we have considered the analysis for the test equation with multiplicative noise only, but not additive. In this note we will study numerical stability analysis for a test system with additive noise, and will show some results for the Euler-Maruyama method.

1. Introduction

Numerical stability for stochastic differential equations (SDEs) has been studied. There should be two types of test equations, with additive noise and multiplicative noise. We have proposed the numerical *mean-square* stability (*MS*-stability) and *trajectory* stability (*T*-stability) for a scalar SDE with one multiplicative noise [7, 6]. However we have not discussed numerical stability for additive noise. In this note we will study numerical stability of the Euler-Maruyama method for a stochastic system with additive noise. Stability analysis for SDE with additive noise can be seen in [2, 3].

Consider the d-dimensional SDE of Ito-type given by

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{f}(t,\boldsymbol{X})\mathrm{d}t + \boldsymbol{g}(t,\boldsymbol{X})\mathrm{d}W(t), \quad t \geq 0, \quad \boldsymbol{X}(0) = \boldsymbol{X}_0 \in \mathbb{R}^n \tag{1}$$

where $\boldsymbol{f}: \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\boldsymbol{g}: \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $W(t)$ is a scalar Wiener process $(t \geq 0)$. A Wiener process is a Gaussian process with the property that

$$\mathbb{E}\left[W(t)\right] = 0, \qquad \mathbb{E}\left[W(t)W(s)\right] = \min\{t,s\}.$$

We assumed that both $\boldsymbol{f}$ and $\boldsymbol{g}$ are sufficiently smooth so that equation (1) has a unique solution. The noise is called additive if $\boldsymbol{g}(t, \boldsymbol{x})$ does not depend on $\boldsymbol{x}$: otherwise it is called multiplicative. Hereafter we shall choose the case of additive noise, i.e.

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{f}(t,\boldsymbol{X})\mathrm{d}t + \boldsymbol{g}(t)\mathrm{d}W(t). \tag{2}$$

Let $\boldsymbol{X}_n$ be the numerical approximation to $\boldsymbol{X}(t_n) = \boldsymbol{X}(nh)$ with constant step-size h. For the SDE (2), the k th component of the Euler-Maruyama method has the form

$$X_{n+1}^k = X_n^k + f^k(t_n, \boldsymbol{X}_n)h + g^k(t_n)\Delta W_n, \tag{3}$$

where ΔW_n stands for the increment of the Wiener process [1, 3].

Now we consider the following linear stochastic test system with one additive noise

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{L}\boldsymbol{X}\mathrm{d}t + \boldsymbol{b}\mathrm{d}W(t), \tag{4}$$

where $\boldsymbol{L}$ is $d \times d$ real matrix, $\boldsymbol{b}$ is d-dimensional real vector and $W(t)$ is a scalar standard Wiener process. The eigenvalues λ_i ($i = 1, 2, \ldots, d$) of $\boldsymbol{L}$ are assumed distinct and satisty $\Re\lambda_i < 0$ ($i = 1, 2, \ldots, d$). Then there exists a non-singular matrix $\boldsymbol{T}$ such that

$$\boldsymbol{T}^{-1}\boldsymbol{L}\boldsymbol{T} = \Lambda := \mathrm{diag}[\lambda_1, \lambda_2, \cdots, \lambda_d].$$

We now define $\boldsymbol{Y}(t) = \boldsymbol{T}^{-1}\boldsymbol{X}(t)$ and on pre-multiplying (4) by $\boldsymbol{T}^{-1}$, we obtain

$$\mathrm{d}\boldsymbol{Y} = \Lambda\boldsymbol{Y}\mathrm{d}t + \boldsymbol{T}^{-1}\boldsymbol{b}\,\mathrm{d}W(t).$$

As for numerical analysis of ordinary differential equations (ODEs), we conclude that the linear stability analysis for the system (4) is reduced to the following scalar test equation

$$\mathrm{d}X = \lambda X\mathrm{d}t + \sigma\mathrm{d}W(t), \quad X(0) = X_0, \quad \lambda, \sigma \in \mathbb{C}, \ \ \Re\lambda < 0. \tag{5}$$

The exact solution of (5) is

$$X(t) = e^{\lambda t}X_0 + \sigma\int_0^t e^{\lambda(t-s)}\mathrm{d}W(s).$$

We observe that

$$\mathbb{E}[X(t)] = e^{\lambda t}\,\mathbb{E}[X_0], \ \ \mathbb{E}[|X(t)|^2] = e^{2\Re\lambda t}\,\mathbb{E}[|X_0|^2] - \frac{|\sigma|^2}{2\Re\lambda}(1 - e^{2\Re\lambda t}).$$

Therefore we have

$$\mathbb{E}[X(t)] \to 0 \tag{6}$$

and

$$\mathbb{E}[|X(t)|^2] \to -\frac{|\sigma|^2}{2\Re\lambda} \tag{7}$$

as $t \to \infty$ from $\Re\lambda < 0$ [5].

2. Stability of Euler-Maruyama method

First we will study numerical stability of the Euler-Maruyama method with respect to mean. Numerical stability in mean is corresponding to the property (6). The Euler-Maruyama method for the test equation (5) is

$$X_{n+1} = X_n + \lambda X_n h + \sigma \Delta W_n . \tag{8}$$

Taking the expectation of each side, we obtain

$$\mathbb{E}\left[X_{n+1}\right] = \mathbb{E}\left[X_n\right](1+\lambda h) = \cdots = \mathbb{E}\left[X_0\right](1+\lambda h)^{n+1} .$$

Thus if $\left|1+\lambda h\right| < 1$, we have

$$\mathbb{E}\left[X_n\right] \to 0 \tag{9}$$

as $n \to \infty$.

Then we can give the following definition for numerical methods of SDEs.

Definition 1 The numerical method is said to be *numerical stable in mean* if the numerical solution X_n for the test equation (5) is satisfied $\mathbb{E}\left[X_n\right] \to 0$ as $n \to \infty$.

For the Euler-Maruyama method, we have the following result.

Theorem 1 For the test equation (5) the Euler-Maruyama method is numerical stable in mean if $\left|1+\lambda h\right| < 1$.

Note that the stability condition $\left|1+\lambda h\right| < 1$ is the same as Euler method for ODEs [4]. In general the numerical stability conditions for the additive case are coincident with them for ODEs [2].

Next we will focus on the numerical solution in mean square. Like as in mean, we get the following relation

$$\mathbb{E}[|X_{n+1}|^2] = \left|1+\lambda h\right|^2 \mathbb{E}[|X_n|^2] + \left|\sigma\right|^2 h .$$

Continuing the iteration,

$$\begin{aligned}
\mathbb{E}[|X_{n+1}|^2] &= \left|1+\lambda h\right|^2 \left\{ |1+\lambda h|^2 \, \mathbb{E}[|X_{n-1}|^2] + \left|\sigma\right|^2 h \right\} + \left|\sigma\right|^2 h \\
&= \left|1+\lambda h\right|^4 \mathbb{E}[|X_{n-1}|^2] + \left\{ |1+\lambda h|^2 + 1 \right\} \left|\sigma\right|^2 h \\
&= \left|1+\lambda h\right|^{2(n+1)} \mathbb{E}[|X_0|^2] + \left\{ |1+\lambda h|^{2n} + \cdots + |1+\lambda h|^2 + 1 \right\} \left|\sigma^2\right| h \\
&= \left|1+\lambda h\right|^{2(n+1)} \mathbb{E}[|X_0|^2] + \frac{\left|1+\lambda h\right|^{2(n+1)} - 1}{\left|1+\lambda h\right|^2 - 1} \left|\sigma\right|^2 h \\
&= \left|1+\lambda h\right|^{2(n+1)} \mathbb{E}[|X_0|^2] + \frac{\left|1+\lambda h\right|^{2(n+1)} - 1}{2\Re\lambda + \left|\lambda\right|^2 h} \left|\sigma\right|^2 .
\end{aligned}$$

If the Euler-Maruyama method is numerical stable in mean for (5), we have the following property of the numerical solution with respect to mean square,

$$\mathbb{E}[|X_n|^2] \to -\frac{|\sigma|^2}{2\Re\lambda + |\lambda|^2 h} \quad (n \to \infty).$$

Note that the equilibrium value in mean square sense is different from the true value $-\frac{|\sigma|^2}{2\Re\lambda}$. However the equilibrium value $\lim_{n\to\infty} \mathbb{E}[|X_n|^2]$ holds the following property

$$-\frac{|\sigma|^2}{2\Re\lambda + |\lambda|^2 h} \to -\frac{|\sigma|^2}{2\Re\lambda} \quad \text{as} \quad h \to 0. \tag{10}$$

This property can be found in Yuan and Mao [8]. Thus we will give the following definition for the asymptotic property in mean square.

Definition 2 The numerical method is said to be *asymptotically consistent in mean square* if the numerical solution $\{X_n\}$ for the test equation(5) satisfies

$$\lim_{h\to 0}(\lim_{n\to\infty} \mathbb{E}[|X_n|^2]) = -\frac{|\sigma|^2}{2\Re\lambda}.$$

We therefore obtain the following result for the Euler-Maruyama method.

Theorem 2 If the Euler-Maruyama method satisfies the numerical stable condition in mean, i.e. $|1+\lambda h| < 1$, the Euler-Maruyama method is asymptotically consistent in mean square.

3. Conclusions and Future aspects

We studied numerical stability for a test system with additive noise and gave some results for the Euler-Maruyama method. We will analyze the other methods, e.g. stochastic theta, Runge-Kutta type, multistep and implicit methods.

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