Weak numerical solution for multiplicative noise stochastic differential equations

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Abstract As for deterministic ordinary differential equations, various numerical schemes are proposed for stochastic differential equations. In this note, we treat weak numerical solutions for stochastic differential equations with multiplicative noise. Some results of the mean error by the simplified weak Euler scheme and its numerical asymptotic stability are shown, and their relation is illustrated through some numerical experiments.

1. Introduction

We consider the Itô scalar stochastic differential equations (SDEs) with multiplicative noise

$$\mathrm{d}X(t) = f(X)\mathrm{d}t + g(X)\mathrm{d}W(t), \quad t \geq 0,\ X(0) = X_0 \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $W(t)$ is a scalar Wiener process($t \geq 0$). A Wiener process is a Gaussian process with the property that

$$\mathbb{E}\left[W(t)\right] = 0, \qquad \mathbb{E}\left[W(t)W(s)\right] = \min\{t, s\}.$$

We assume that both f and g are sufficiently smooth so that equation (1) has a unique solution. We will treat weak numerical solutions (the first two moments) for (1) in this note.

For the SDE (1), Euler-Maruyama scheme has the form

$$X_{n+1} = X_n + f(X_n)h + g(X_n)\Delta W_n \tag{2}$$

where h and ΔW_n stand for the step-size and the increment of the Wiener process, respectively. The Euler-Maruyama scheme (2) has order of weak convergence 1[4]. In weak approximation, it is known that the random increments ΔW_n of the Wiener process can be replaced by other convenient approximations $\Delta \hat{W}_n$ which have similar moment properties to the ΔW_n [4]. For example we could use the two-point distributed random variable

$$P\left(\Delta \hat{W}_n = \pm\sqrt{h}\right) = \frac{1}{2}$$

for the Euler-Maruyama scheme (2). This leads to the **simplified weak Euler scheme**

$$X_{n+1} = X_n + f(X_n)h + g(X_n)\Delta \hat{W}_n. \qquad (3)$$

Now we will estimate mean error by simplified weak Euler scheme through some numerical experiments. We test the following example whose means is submartingale (mean increasing).

Example 1

$$\mathrm{d}X(t) = \frac{1}{2} X\,\mathrm{d}t + \sigma X\,\mathrm{d}W(t), \quad t \in [0, T], \quad X(0) = 1. \qquad (4)$$

The exact solution of the equation (4) is

$$X(t) = \exp\left(\frac{1}{2}\left(1 - \sigma^2\right) t + \sigma W(t) \right).$$

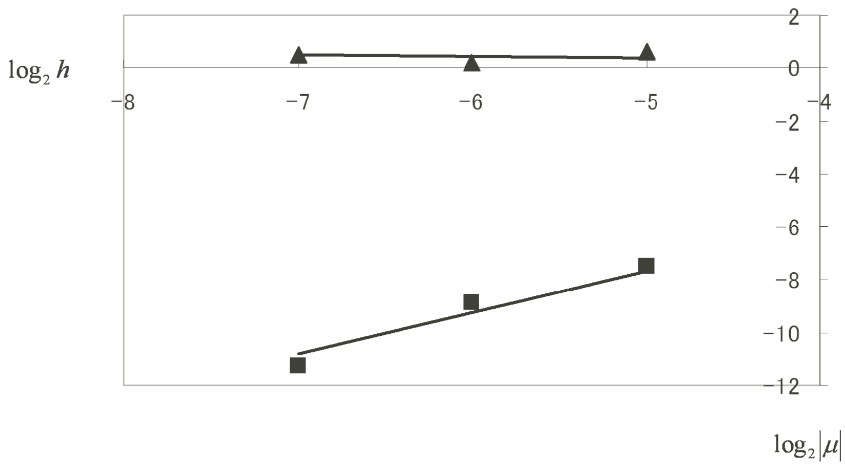
Also the expectation of the solution of (4) is

$$\mathbb{E}\left[X(t)\right] = \exp\left(\frac{1}{2} t \right).$$

We shall compute the k-th trajectory $X_{T,k,h}$ of simplified weak Euler scheme (3) with step-size h and estimate the mean error with N trajectories at T by

$$\mu = \frac{1}{N} \sum_{k=1}^{N} X_{T,k,h} - \mathbb{E}\left[X(T)\right].$$

We chose $T = 1.0$, $N = 20000$, and step-size $h = 2^{-5}, 2^{-6}, 2^{-7}$. We show the results of Example 1 for two cases (i) $\sigma = 0.1$ and (ii) $\sigma = 5$ in Figure 1.



■: $\sigma = 0.1$ ▲: $\sigma = 5$

Figure 1: Mean error by simplified weak Euler scheme for Example 1.

From Figure 1 we can see that the $\log_2$ of the mean error closely follows a straight line of slope 1 in $\log_2 h$ for (i) $\sigma = 0.1$. However the result of (ii) $\sigma = 5$ indicates that it does not converge. Namely, it is not appear to be the rate of convergence of the simplified weak Euler scheme for (ii). In fact this phenomenon is deeply related to the asymptotic stability in SDEs. We will describe the numerical asymptotic stability in the following section.

2. Numerical asymptotic stability

We shall briefly introduce the notion of numerical asymptotic stability in this section. Consider the following scalar linear test equation,

$$\mathrm{d}X(t) = \lambda X\,\mathrm{d}t + \sigma X\,\mathrm{d}W(t), \quad t \geq 0, \quad \lambda, \sigma \in \mathbb{R} \tag{5}$$

with the initial condition $X(0) = 1$. Since the exact solution of (5) is

$$X(t) = \exp\left(\left(\lambda - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right),$$

we can see that the equilibrium position $X(t) \equiv 0$ is stochastically asymptotically stable if $\lambda - \frac{1}{2}\sigma^2 < 0$ [1]. For Example 1, the case (ii) $\sigma = 5$ is stochastically asymptotically stable, wheres the case (i) $\sigma = 0.1$ is not stochastically asymptotically stable.

We shall give the following definition which is found in [2,3].

Definition 1 *When a numerical scheme is applied to the asymptotically stable equation (5) and generating the sequence* $\{X_n\}$*, it is said to be numerically asymptotically stable if*

$$\lim_{n\to\infty} |X_n| = 0 \quad \text{w.p.1}.$$

We proposed the notion of T-stability for a numerical scheme with respect to its trajectory [6].

Definition 2 *Assume that the test equation (5) is stochastically asymptotically stable. The numerical scheme equipped with a specified driving process said to be T-stable if*

$$\lim_{n\to\infty} |X_n| = 0 \quad \text{w.p.1}.$$

holds for the driving process.

Now we will seek for T-stable condition of simplified weak Euler scheme (3). The averaged stability function $T(h;\lambda,\sigma)$ of the scheme for the test equation (5) is

$$T(h;\lambda,\sigma)=\sqrt{\left|(1+\lambda h)^2-\sigma^2 h\right|},$$

and the necessary and sufficient condition for T-stability is $T(h;\lambda,\sigma)<1$ [6].

For Example 1 (ii) $\lambda=\frac{1}{2}$ and $\sigma=5,$ we have

$$\left|\left(1+\frac{1}{2}h\right)^2-25\,h\right|<1.$$

We finally get the interval of step-size h

$$0<h<48-2\sqrt{574},\quad 48+2\sqrt{574}<h<96.$$

We adopt the interval $0<h<48-2\sqrt{574}\approx 0.083$ because of its weak convergence. Note that we chose to the values of step-size are smaller than 0.083 in the Example 1 (ii) σ=5.

Now we can see the phenomenon in Example 1 (ii) by the long numerical time-integration for it. We shall compute the arithmetic mean $\overline{X}_n$ at $t=nh$,namely

$$\overline{X}_n=\frac{1}{N}\sum_{k=1}^{N}X_{n,k,h}$$

with $h=2^{-5}$ and N=20000 for Example 1 (ii). The result is shown in Figure 2.

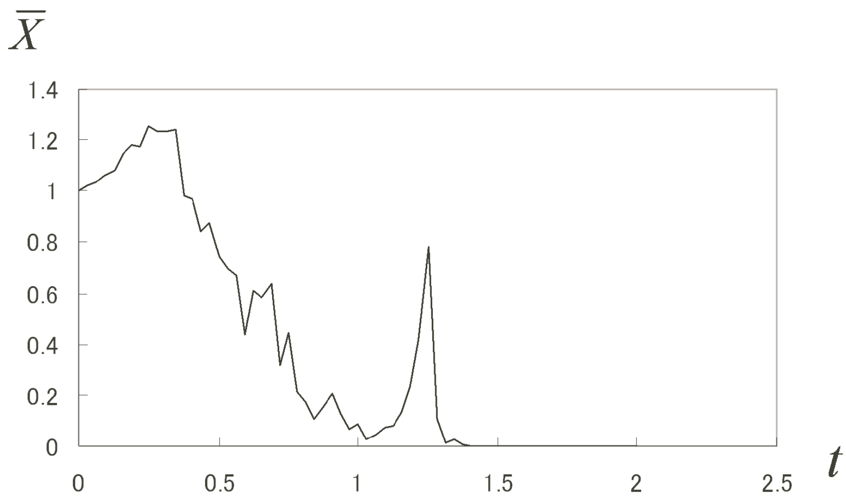


Figure 2: Example 1 (ii) $\sigma=5, h=2^{-5}$ and N=20000.

We therefore obtain the following result for the asymptotic stability of the arithmetic mean $\overline{X}_n$.

Proposition 1 *If the numerical solution* X_n *for the test equation (5) is numerically asymptotically stable, then*

$$\lim_{n\to\infty}\left|\overline{X}_n\right| = 0 \quad \text{w.p.1.}$$

Note that we can get the same result for the mean-square value of the numerical solution for the test equation (5) [7].

3. Conclusions

We described that weak numerical solution for SDEs with multiplicative noise is influenced by its asymptotic numerical stability in the present note. To avoid the incorrect numerical results of weak numerical schemes, we proposed the another type of error analysis [5]. It could be solved by separating mean error μ into two parts (deterministic and stochastic), namely

$$\mu = \mu_{\text{det}} + \mu_{\text{sto}}.$$

Here

$$\mu = \frac{1}{N}\sum_{k=1}^{N} X_{T,k,h} - \mathbb{E}\left[X(T)\right],$$

$$\mu_{\text{det}} = \frac{1}{N}\sum_{k=1}^{N} X_{T,k,h} - \frac{1}{N}\sum_{k=1}^{N} \widetilde{X}_{T,k,h},\ \mu_{\text{sto}} = \frac{1}{N}\sum_{k=1}^{N} \widetilde{X}_{T,k,h} - \mathbb{E}\left[X(T)\right].$$

$\widetilde{X}_{T,k,h}$ is the time-discrete realized solution at T with step-size h for k-th trajectory. For the Example 1, $\widetilde{X}_{T,k,h}$ has the following form

$$\widetilde{X}_{T,k,h} = \exp\left(\frac{1}{2}\left(1-\sigma^2\right)T + \sigma \hat{W}_{T,k,h}\right),$$

where $\hat{W}_{T,k,h}$ is the discretized Wiener driving process of step-size h, namely

$$\hat{W}_{T,k,h} = \sum_{i=0}^{m-1} \Delta\hat{W}_{i,k}, \quad m = \frac{h}{T}.$$

Note that $\Delta\hat{W}_{i,k}$ is the same random variable used in simplified weak Euler scheme (3). We called $\widetilde{X}_{T,k,h}$ the realized solution, μ_{det} as deterministic part, μ_{sto} as stochastic part. We anticipate that the rate of convergence of the scheme appears in deterministic part, the sampling error and the influence of asymptotic stability appear in stochastic part. We show the numerical result of $\mu_{\det}$ for Example 1 in Figure 3. Figure 3 indicates that the rate of convergence of weak Euler scheme is 1 for both (i) and (ii).

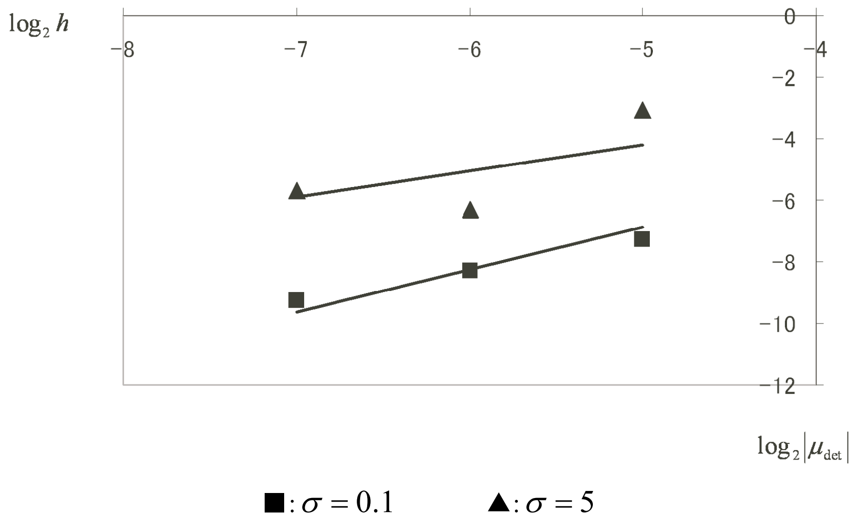


Figure 3: $\mu_{\det}$ of simplified weak Euler scheme for Example 1.

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