

Non-trivial Torus Equivariant Vector bundles of Rank Three on \mathbf{P}^2

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Abstract

Let T be a two dimensional algebraic torus over an algebraically closed field k . Then \mathbf{P}^2 has a non trivial action of T and becomes a toric variety. Let E be a torus equivariant vector bundle on \mathbf{P}^2 . Since the restriction of an equivariant vector bundle to an affine toric variety is trivial, we can find the semi-invariant bases. These bases and patching data make numerical data. I have already classified equivariant vector bundles on a non-singular toric variety using these numerical data. In this paper we study non trivial equivariant vector bundles of rank three on \mathbf{P}^2 . Numerical data are good for calculations by a computer. So I have tried to calculate these numerical data by a computer and I have gotten the result of this paper.

§ 1 Torus equivariant vector bundles

Let N be a free \mathbf{Z} -module of rank n . Let M be the dual \mathbf{Z} -module of N . Then there is a natural \mathbf{Z} -bilinear map

$$\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbf{Z}.$$

It can naturally be extended to $M_{\mathbf{R}} \times N_{\mathbf{R}} \rightarrow \mathbf{R}$, where $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$ and $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$. We denote $\varphi(\xi) = \langle \xi, \varphi \rangle$ for ξ in $M_{\mathbf{R}}$ and φ in $N_{\mathbf{R}}$. Let $T = T_N$ be an n -dimensional algebraic torus defined by N over an algebraically closed field k . Then we can identify M with the additive group of characters of T . Let the exponential map $e: M \rightarrow k(T)^*$ be the homomorphism which sends ξ in M to the corresponding rational function $e(\xi)$ on T .

We call a non-empty subset C of $N_{\mathbf{R}}$ a strongly convex polyhedral cone with apex at $\mathbf{0}$ or simply a cone, if $C \cap (-C) = \{0\}$ and if there exists a finite subset $\{\varphi_1, \dots, \varphi_m\}$ of $N_{\mathbf{R}}$ such that $C = \mathbf{R}_0\varphi_1 + \dots + \mathbf{R}_0\varphi_m$ where \mathbf{R}_0 denotes the set of non-negative real numbers. Let the dimension of C be the dimension of the \mathbf{R} -vector space $C + (-C)$.

A non-empty subset C' of a cone C is called a facial cone of C if there exists an element ξ of M such that $\varphi(\xi) \geq 0$ for all φ in C and that $C' = \{\varphi \in C \mid \varphi(\xi) = 0\}$.

By a fan Σ in N is meant a finite set of cones C in $N_{\mathbf{R}}$ such that

- (i) if C' is a facial cone of C in Σ then C' is also a cone in Σ ,
- (ii) if C and C' are two cones in Σ then the intersection $C \cap C'$ is a facial cone of C as well as of C' .

For a fan Σ , the toric variety $X = X(\Sigma)$ is constructed by gluing the affine toric varieties U_C where C in Σ .

A toric variety is an algebraic variety X over k endowed with an action of T and which has a dense orbit. Normal effective toric varieties under torus action have been classified.

DEFINITION 1. 1. An equivariant vector bundle E on a non-singular toric variety X such that there exists an isomorphism $f_t : t^*E \rightarrow E$ for every k -rational point t in T where $t : X \rightarrow X$ is the action of t on X .

DEFINITION 1. 2. An equivariant vector bundle $E = (E, f_t)$ is said to be T -linearized if $f_{st} = f_t \circ t^*f_s$ holds for every pair of k -rational points s, t in T , where

$$f_{st} = f_t \circ t^*f_s : (st)^*E \xrightarrow{t^*f_s} t^*E \xrightarrow{f_t} E.$$

In [3] we showed that an equivariant vector bundle necessarily has a T -linearization. We also studied how to describe T -linearized vector bundle in terms of fans, as we now recall. Let Σ be a fan of N and we denote $\Sigma(l)$ be the set of an l -dimensional cones. For C in $\Sigma(l)$, there exists a finite subset $\{\varphi_1, \dots, \varphi_l\}$ of N and $C = \mathbf{R}_0\varphi_1 + \dots + \mathbf{R}_0\varphi_l$ where \mathbf{R}_0 is the set of non-negative real numbers. We say that $\{\varphi_1, \dots, \varphi_l\}$ is the fundamental system of generators of C if φ_i ($1 \leq i \leq l$) are primitive. The fundamental system of generators $\{\varphi_1, \dots, \varphi_l\}$ of C is uniquely determined by C and is denoted by $|C|$. We consider the following:

$$(i) \ m : \{|C| \mid C \in \Sigma(1)\} \longrightarrow \mathbf{Z}^{\oplus r}$$

sending φ to $m(\varphi) = (m(\varphi)_1, \dots, m(\varphi)_r)$, and there is a map for every C in $\Sigma(n)$,

$$m_C : |C| \longrightarrow \mathbf{Z}^{\oplus r}$$

so that there exists a permutation $\tau = \tau_C$ such that

$$m_C(\varphi) = (m_C(\varphi)_1, \dots, m_C(\varphi)_r) = (m(\varphi)_{\tau(1)}, \dots, m(\varphi)_{\tau(r)})$$

for every φ in $|C|$.

Let C be an n -dimensional cone in $\Sigma(n)$. Then we have a set of characters $\{\xi(C)_1, \dots, \xi(C)_r\}$ in M by solving the equations $\varphi(\xi(C)_i) = m_C(\varphi)_i$ for every φ in $|C|$ ($1 \leq i \leq r$). Then it is easy to see that (i) is equivalent to the following:

$$(i) \ \xi : \Sigma(n) \longrightarrow M^{\oplus r}$$

sending C to $\xi(C) = (\xi(C)_1, \dots, \xi(C)_r)$ such that there exists a permutation $\tau = \tau_{C,C'}$ for every pair of cones C and C' in $\Sigma(n)$ so that $\varphi(\xi(C)_i) = \varphi(\xi(C')_{\tau(i)})$ for every i and every φ in $|C| \cap |C'|$.

$$(ii) \ P : \Sigma(n) \times \Sigma(n) \longrightarrow GL_r(k)$$

sending (C, C') to $P(C, C') = (P(C, C')_{ij})$ such that (i, j) -component $P(C, C')_{ij} \neq 0$ only if $\varphi(\xi(C)_i) \geq \varphi(\xi(C')_j)$ for every φ in $|C| \cap |C'|$ and that

$$P(C, C')P(C', C'') = P(C, C'')$$

for every C, C', C'' in $\Sigma(n)$.

For a data (m, P) defined by (i) and (ii), we denote by $E(m, P)$ the T -linearized vector bundle obtained from the data (m, P) . Now we consider the condition that two vector bundles $E(m, P)$ and $E(m', P')$ are T -isomorphic. We can describe these condition using these data.

- (iii) Two pairs (m, P) and (m', P') defined by (i) and (ii) are said to be equivalent if there exists a permutation $\tau = \tau_C$ for every C in $\Sigma(n)$ such that

$$(m_C(\varphi)_1, \dots, m_C(\varphi)_r) = (m'_{C'}(\varphi)_{\tau(1)}, \dots, m'_{C'}(\varphi)_{\tau(r)})$$

for every φ in $|C|$ and if there exists

$$\sigma : \Sigma(n) \longrightarrow GL_r(k)$$

such that (i, j) -component $\sigma(C)_{ij} \neq 0$ only if $\varphi(\xi(C)_i) \geq \varphi(\xi(C')_j)$ for every φ in $|C|$ and such that

$$P'(C, C') = \sigma(C)^{-1}P(C, C')\sigma(C')$$

hold for every C and C' in $\Sigma(n)$.

THEOREM 1. 3. Let $X = X(\Sigma)$ be a smooth complete toric variety defined by a fan Σ . Then the set of T -linearized vector bundle of rank r up to T -isomorphism corresponds bijectively to the set of (i) (or (i)) and (ii) up to equivalence (iii).

§ 2 Equivariant vector bundles on \mathbf{P}^2

Let $X = \mathbf{P}^2$ be a two dimensional projective space. This has a natural two dimensional torus T action and it becomes a toric variety. At first we consider equivariant vector bundles of rank 2 on \mathbf{P}^2 .

DEFINITION 2. 1. Let $E = E(a, b, c)$ be a vector bundle defined for positive integers a, b, c by the following exact sequence,

$$0 \longrightarrow O_{\mathbf{P}^2} \xrightarrow{f} O_{\mathbf{P}^2}(a) \oplus O_{\mathbf{P}^2}(b) \oplus O_{\mathbf{P}^2}(c) \longrightarrow E(a, b, c) \longrightarrow 0.$$

The map f is defined by sending 1 to (X_0^a, X_1^b, X_2^c) where (X_0, X_1, X_2) is the homogeneous coor-

dinates of \mathbf{P}^2 .

This vector bundle $E(a, b, c)$ for positive integers a, b, c is an indecomposable equivariant vector bundle of rank 2 on \mathbf{P}^2 . We can calculate the data (m, P) of the equivariant vector bundle $E(a, b, c)$.

THEOREM 2.2. Let E be an indecomposable equivariant vector bundle of rank 2 on \mathbf{P}^2 . Then E is isomorphic to $E(a, b, c) \otimes \mathcal{O}_{\mathbf{P}^2}(n)$ for some integer n and positive integers a, b, c .

From now on, we consider the equivariant vector bundles of rank 3 on \mathbf{P}^2 . Let $\{\varphi_1, \varphi_2\}$ be a \mathbf{Z} -base of N and put $\varphi_0 = -\varphi_1 - \varphi_2$ and

$$C = \mathbf{R}_0\varphi_1 + \mathbf{R}_0\varphi_2, \quad C' = \mathbf{R}_0\varphi_1 + \mathbf{R}_0\varphi_0, \quad C'' = \mathbf{R}_0\varphi_0 + \mathbf{R}_0\varphi_1.$$

Let Σ be a fan defined by $\{\varphi_0, \varphi_1, \varphi_2\}$, then $\Sigma(2) = \{C, C', C''\}$. This fan Σ defines the toric variety \mathbf{P}^2 . Put

$$\begin{aligned} m_C(\varphi_1) &= m_{C'}(\varphi_1) = (a_1, a_2, a_3), \\ m_C(\varphi_2) &= m_{C'}(\varphi_2) = (b_1, b_2, b_3), \\ m_{C'}(\varphi_0) &= (c_1, c_2, c_3), \\ m_{C''}(\varphi_0) &= (d_1, d_2, d_3). \end{aligned}$$

where a_i, b_i, c_i, d_i ($i = 1, 2, 3$) are integers and the two sets $\{c_1, c_2, c_3\}$ and $\{d_1, d_2, d_3\}$ are the same set. Now we calculate the matrices $P(C', C), P(C, C''), P(C', C'')$ such that the corresponding equivariant vector bundle is indecomposable. We have gotten the following table in which the first line means the order of $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$ and (d_1, d_2, d_3) such that s, m and l ($s < m < l$) mean the order of them and the second line means the matrices $P(C', C), P(C, C'')$, and $P(C', C'')$.

THEOREM 2.3. Let $E = E(m, P)$ for the data (m, P) be an indecomposable equivariant vector bundle of rank three on \mathbf{P}^2 . Then the data (m, P) is one of the following table.

41: (l,m,s) (s,m,l) (m,l,s) (s,m,l)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
42: (l,m,s) (s,m,l) (l,s,m) (s,m,l)	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$
43: (l,m,s) (s,m,l) (l,m,s) (s,m,l)	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
44: (l,m,s) (s,m,l) (l,m,s) (s,m,l)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
45: (l,m,s) (s,m,l) (l,m,s) (s,l,m)	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
46: (l,m,s) (s,m,l) (l,m,s) (m,s,l)	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
47: (l,m,s) (s,l,m) (l,s,m) (s,m,l)	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$
48: (l,m,s) (m,s,l) (m,l,s) (s,m,l)	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Next we calculate the data such that the equivariant vector bundle E is decomposed into $E = L \oplus F$ where L is a line bundle and F is an indecomposable vector bundle of rank two. Then we have gotten the following theorem.

THEOREM 2. 4. Let $E = E(m, P)$ for the data (m, P) be an equivariant vector bundle of rank three on \mathbf{P}^2 . Suppose that E decomposes into $E = L \oplus F$ where L is a line bundle and F is an indecomposable vector bundle. Then the data (m, P) is one of the following table.

141: (l,m,s) (l,s,m) (s,m,s) (s,s,m)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
142: (l,m,s) (l,s,m) (s,l,m) (s,m,l)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
143: (l,m,s) (l,s,m) (m,m,s) (m,s,m)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
144: (l,m,s) (l,s,m) (m,l,s) (m,s,l)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
145: (l,m,s) (l,s,m) (l,m,s) (l,s,m)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

References

- [1] G. Ewald: Combinatorial convexity and algebraic geometry. G.T.M., Springer Verlag (1996)
- [2] W. Fulton: Introduction to toric varieties. Ann. Math. Studies, Princeton University Press (1993)
- [3] T. Kaneyama: On equivariant vector bundles on an almost homogeneous variety. Nagoya Math. J. 57 (1975) 65-86.
- [4] T. Kaneyama: Torus-equivariant vector bundles of rank three on P^2 . The Ann. of Gifu Univ. of Education. 14 (1987) 47-55.
- [5] T. Kaneyama: Torus-equivariant vector bundles on projective space. Nagoya Math. J. 111 (1988) 25-40.
- [6] T. Kaneyama: Some Properties of torus equivariant vector bundles. The Ann. of Gifu Univ. of Education. 20 (1990) 127-133.
- [7] T. Kaneyama: Some Properties of torus equivariant vector bundles II. The Ann. of Gifu Univ. for Education and Languages. 33 (1997) 223-230.
- [8] T. Oda: Torus embeddings and applications. Tata Institute of Fundamental Research 57 (1978)
- [9] T. Oda: Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Ergeb. der Math., Springer Verlag (1988)