Jacobian determinant and system of parameters

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Abstract

We characterize systems of parameters of a formal power series ring by their Jacobian determinants.

Introduction

Jacobian determinants appear in many areas of mathematics and play important roles. Commutative ring theory is not an exception. In this paper R denotes a formal power series ring $k[[x_1, x_2, \ldots, x_n]]$ over a field $k$ of characteristic 0 and $m$ denotes the maximal ideal of $R$. A subset $(f_1, \ldots, f_n)$ of $m$ is called a system of parameters of $R$ if $R/(f_1, \ldots, f_n)$ is a finite dimensional vector space over $k$. The purpose of this paper is to characterize a system of parameters of $R$ by its Jacobian determinant with respect to $x_1, x_2, \ldots, x_n$.

Preliminaries

We begin with recalling the definition of a Jacobian matrix and a Jacobian determinant. Let $f_1, \ldots, f_n \in R$ then the $r \times m$ matrix $(\frac{\partial f_i}{\partial x_j})$ is called the Jacobian matrix of $f_1, \ldots, f_n$ with respect to $x_1, \ldots, x_n$ and $\det(\frac{\partial f_i}{\partial x_j})$ is called the Jacobian determinant of $f_1, \ldots, f_n$ with respect to $x_1, \ldots, x_n$. We refer the reader to [M] for fundamental properties of Jacobian matrices and determinants.

Lemma 1. If $f_1, \ldots, f_n$ form a system of parameters of $R$ then $(f_1, \ldots, f_n)m=(f_1, \ldots, f_n)\det(\frac{\partial f_i}{\partial x_j})$, where $\det(\frac{\partial f_i}{\partial x_j})$ is the Jacobian determinant of $f_1, \ldots, f_n$ with respect to $x_1, \ldots, x_n$ and $\det(\frac{\partial f_i}{\partial x_j})\neq0$. We have $det(\frac{\partial f_i}{\partial x_j})\in Q$.

Proof. See [K] Appendices F.

Lemma 2. Let $Q$ be an $m$-primary ideal of $R$ which is not a complete intersection. For any system of parameters $f_1, \ldots, f_n \in Q$ we have $\det(\frac{\partial f_i}{\partial x_j})\in Q$.

Proof. Since $(f_1, \ldots, f_n)$ is an $m$-primary ideal of $R$ and $(f_1, \ldots, f_n)\supseteq Q$ there is an integer $n$ such that $m^{n+1}Q \supseteq (f_1, \ldots, f_n)$ and $m^nQ \supseteq (f_1, \ldots, f_n)$. Take an element $a \in m^nQ$ such that $a \in (f_1, \ldots, f_n)$ Then $ma \supseteq (f_1, \ldots, f_n)$. Therefore $(f_1, \ldots, f_n)m \supseteq Q(f_1, \ldots, f_n)$. By Lemma 1, $(f_1, \ldots, f_n)m/(f_1, \ldots, f_n)$ is a vector space of dimension 1 over $k$. Hence $(f_1, \ldots, f_n)m \supseteq Q(f_1, \ldots, f_n)$. By Lemma 1, we have $\det(\frac{\partial f_i}{\partial x_j})\in Q$.

Characterization of complete intersection.

The following theorem is the main result of this paper.

Theorem 3. If $f_1, \ldots, f_n \in m$ satisfy $\det(\frac{\partial f_i}{\partial x_j})\neq0$ if $f_i, \ldots, f_n$ then $f_1, \ldots, f_n$ form a system of parameters of $R$.
To prove this theorem we need three more lemmas.

**Lemma 4.** Let $Q = (g_1, \square, g_d)$ be an $m$-primary ideal and $f_1, \square, f_d \subseteq Q$. If $\det(\partial f_i/\partial x_j) \in Q$ then $Q = (f_1, \square, f_d)$

**Proof.** We can write $f_i = a_i g_i + \square + a_0 g_d$. By differentiating with respect to $x_i, \square, x_d$ we have

$$\partial f_i/\partial x_j = a_i \partial g_i/\partial x_j + \square + a_0 \partial g_d/\partial x_j \mod Q.$$  

Then, we get $\det(\partial f_i/\partial x_j) \equiv \det(a_i) \det(\partial g_i/\partial x_j) \mod Q$. If $\det(a_i) \equiv m$ then $\det(\partial f_i/\partial x_j) \in Q$, by Lemma 1. Therefore, if $\det(\partial f_i/\partial x_j) \equiv Q$ we have $\det(a_i) \equiv m$. Hence, the matrix $(a_i)$ is invertible and therefore $Q = (f_1, \square, f_d)$

**Lemma 5.** Let $p$ be a prime ideal of $R$ such that $p \neq m$ and let $K = R_{P}/R$. If $htp = r$ and $x_1, \square, x_d$ form a system of parameters of $R/P$ then $(R_{p})^{\square}[y_1, \square, y_j]$ and $(\partial f_i/\partial x_j)_{\square} = (\partial f_i/\partial x_j)_{\square} A$ for some invertible matrix $A$ over $(R_{p})^{\square}$ where $(R_{p})^{\square}$ is the completion of $R_{p}$.

**Proof.** $\partial f_i/\partial x_j$ can be extended to a $K$-derivation of $(R_{p})^{\square}$. We can write $\partial f_i/\partial x_j = a_i \partial f_i/\partial y_j + \square + a_0 \partial f_i/\partial y_j$, for some $a_0 \subseteq (R_{p})^{\square}$. Since $\partial f_i/\partial x_j = \partial f_i/\partial x_j$ the matrix $A = (a_i)$ is invertible.

**Lemma 6.** Let $q$ be a primary ideal such that $\sqrt{q} \neq m$. If $f_1, \square, f_d \subseteq q$ then $\det(\partial f_i/\partial x_j) \subseteq q$.

**Proof.** Let $p = \sqrt{q}$. We separate two cases.

**Case 1.** $q_{R_p}$ is a complete intersection. If $htp = r$ then there are $y_1, \square, y_j \subseteq p$ such that $p_{R_p} = (y_1, \square, y_j)$, $R_p$. Then the $p_{R_p}$-adic completion $(R_p)$ is isomorphic to $K[y_1, \square, y_j]$ where $K = R_p/p_{R_p}$. We can choose $x_1, \square, x_d$ so that $x_1, \square, x_d$ form a system of parameters of $R/P$.

Then, for $1 \leq j \leq r$, the $k$-derivation $\partial f_i/\partial x_j$ can be extended to a $K$-derivation of $K[y_1, \square, y_j]$. Suppose that $\det(\partial f_i/\partial x_j) \subseteq q$. Then we can assume that $\det(\partial f_i/\partial x_j) \subseteq q$. By Lemma 5. Then by Lemma 4,

$(f_1, \square, f_d, R_p) = q_{R_p}$

There is an $s \subseteq R_{p}$ such that

$s_t \equiv (f_1, \square, f_d) \mod (R_{p})^{\square}$

Then $s_{1}^{\square} \det(\partial f_i/\partial x_j) \equiv 0 \mod q$.

This implies $\det(\partial f_i/\partial x_j) \subseteq q$, a contradiction.

Therefore $\det(\partial f_i/\partial x_j) \subseteq q$ in this case.

**Case 2.** $q_{R_p}$ is not a complete intersection.

We choose generators $g_1, \square, g_d$ of $q_{R_p}$ so that any $r$ of $g_1, \square, g_d$ form a system of parameters of $(R_{p})^{\square}$. Since $\det(\partial f_i/\partial x_j)$ is an $(R_{p})^{\square}$-linear combination of the elements of the form $\det(\partial h_i/\partial x_j)_{x_1, \square, x_d}$, where $h_1, \square, h_d$ is a system of parameters of $(R_{p})^{\square}$ contained in $q_{(R_p)}$

By Lemma 2, we have $\det(\partial f_i/\partial x_j) \subseteq q$.

Now we can prove Theorem 3.

**Proof of Theorem 3.**
Suppose that $\text{ht}(f_i, \emptyset, f_i) < d$.

If $(f_i, \emptyset, f_i) = q_i \cap \emptyset \cap \emptyset \cap Q$ is a primary decomposition of $(f_i, \emptyset, f_i)$ where $Q$ is primary to $m$ then $\det(\partial f_i/\partial x_j) \in q_k(1 \leq k \leq m)$ by Lemma 6 and, by Lemma 7 and Lemma 4 we have $\det(\partial^2 f_i/\partial x_j) \subseteq Q$.

**Corollary 6.** Let $f \in m^2$. Then $R/fR$ has an isolated singularity if and only if the Hessian $\det(\partial^2 f/\partial x_i \partial x_j)$ is not contained in $(\partial f/\partial x_i, \emptyset, \partial f/\partial x_j)$.

**References**

