Jacobian determinant and system of parameters

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Abstract

We characterize systems of parameters of a formal power series ring by their Jacobian determinants.

Introduction

Jacobian determinants appear in many areas of mathematics and play important roles. Commutative ring theory is not an exception. In this paper R denotes a formal power series ring k $[[x_1, x_2, \cdots, x_d]]$ over a field k of characteristic 0 and m denotes the maximal ideal of R. A subset $\{f_1, \cdots, f_d\}$ of m is called a system of parameters of R if $R/(f_1, f_2, \ldots, f_d)$ is a finite dimensional vector space over k. The purpose of this paper is to characterize a system of parameters of R by its Jacobian determinant with respect to $x_1, x_2, \cdots, x_d$.

Preliminaries

We begin with recalling the definition of a Jacobian matrix and a Jacobian determinant. Let $f_1, \cdots, f_r \in R$ then the $r \times d$ matrix $(\partial f_i/\partial x_j)$ is called the Jacobian matrix of $f_1, \ldots, f_r$ with respect to $x_1, \cdots, x_d$ and det $(\partial f_i/\partial x_j)$ is called the Jacobian determinant of $f_1, \cdots, f_r$ with respect to $x_1, \cdots, x_d$. We refer the reader to [M] for fundamental properties of Jacobian matrices and determinants.

Lemma 1. If $f_1, \cdots, f_d$ form a system of parameters of R then $(f_1, \cdots, f_d):m = (f_1, \cdots, f_d \det(\partial f_i/\partial x_j))$, where det $(\partial f_i/x_j)$ is the Jacobian determinant of $f_1, \cdots, f_d$ with respect to $x_1, \cdots, x_d$ and det $(\partial f_i/\partial x_j) \notin (f_1, \cdots, f_d)$.

Proof. See [K], Appendices F.

Lemma 2. Let Q be an m-primary ideal of R which is not a complete intersection. For any system of parameters $f_1, \cdots, f_d \in Q$ of R we have det $(\partial f_i/\partial x_j) \in Q$.

Proof. Since $(f_1, \cdots, f_d)$ is an m-primary ideal of R and $(f_1, \cdots, f_d) \subsetneqq Q$ there is an integer n such that $m^{n+1}Q \subset (f_1, \cdots, f_d)$ and $m^nQ \not\subset (f_1, \cdots, f_d)$. Take an element $a \in m^nQ$ such that $a \notin (f_1, \cdots, f_d)$. Then $ma \subset (f_1, \cdots, f_d)$. Therefore $((f_1, \cdots, f_d):m) \cap Q/(f_1, \cdots, f_d) \neq 0$. By Lemma 1, $(f_1, \cdots, f_d):m/(f_1, \cdots, f_d)$ is a vector space of dimension 1 over k. Hence $((f_1, \cdots, f_d):m) \cap Q/(f_1, \cdots, f_d) = (f_1, \cdots, f_d):m/(f_1, \cdots, f_d)$. By Lemma 1 , we have det $(\partial f_i/\partial x_j) \in Q$.

Characterization of complete intersection.

The following theorem is the main result of this paper.

Theorem 3. If $f_1, \cdots, f_d \in m$ satisfy det $(\partial f_i/\partial x_j) \notin (f_1, \cdots, f_d)$ then $f_1, \cdots, f_d$ form a system of parameters of R.

To prove this theorem we need three more lemmas.

Lemma 4. Let $Q=(g_1, \cdots, g_d)$ be an m-primary ideal and $f_1, \cdots, f_d \in Q$. If $\det(\partial f_i/\partial x_j) \notin Q$ then $Q=(f_1, \cdots, f_d)$.

Proof. We can write $f_j = a_{1j}g_1 + \cdots + a_{dj}g_d$. By differentiating with respect to $x_1, \cdots, x_d$ we have

$$\partial f_j/\partial x_k \equiv a_{1j}\partial g_1/\partial x_k + \cdots + a_{dj}\partial g_d/\partial x_k \bmod Q.$$

Then, we get $\det(\partial f_j/\partial x_j) \equiv \det(a_{ij})\det(\partial g_i/\partial x_j) \bmod Q$.
If $\det(a_{ij}) \in m$ then $\det(\partial f_i/\partial x_j) \in Q$, by Lemma 1. Therefore, if $\det(\partial f_i/\partial x_j) \notin Q$ we have $\det(a_{ij}) \in m$. Hence, the matrix (a_{ij}) is invertible and therefore $Q=(f_1, \cdots, f_d)$.

Lemma 5. Let p be a prime ideal of R such that $p \neq m$ and let $K=R_P/_PR_P$. If $\mathrm{ht}\,p = r$ and $x_{r+1}, \cdots, x_d$ form a system of parameters of R/P then $(R_p) \xrightarrow{\sim} K[[y_1, \cdots, y_r]]$ and $(\partial/\partial x_1, \cdots, \partial/\partial x_r) = (\partial/\partial y_1, \cdots, \partial/\partial y_r)A$ for some invertible matrix A over (R_p), where (R_p) is the completion of R_p.

Proof. $\partial/\partial x_i$ can be extended to a K-derivation of (R_p). We can write $\partial/\partial x_i = a_{1i}\partial/\partial y_1 + \cdots + a_{ri}\partial/\partial y_r$ for some $a_{ki} \in (R_p)$. Since $\partial x_j/\partial x_i = \delta_{ij}$ the matrix $A=(a_{ij})$ is invertible.

Lemma 6. Let q be a primary ideal such that $\sqrt{q} \neq m$. If $f_1, \cdots, f_d \in q$ then $\det(\partial f_i/\partial x_j) \in q$.

Proof. Let $p=\sqrt{q}$. We separate two cases.
Case 1. qR_p is a complete intersection. If $\mathrm{ht}\,p = r$ then there are $y_1, \cdots, y_r \in p$ such that $pR_p = (y_1, \cdots, y_r)R_p$. Then the pR_p-adic completion (R_p) is isomorphic to $K[[y_1, \cdots, y_r]]$ where $K=R_p/pR_p$. We can choose $x_1, \cdots, x_d$ so that $x_{r+1}, \cdots, x_d$ form a system of parameters of R/p.

Then, for $1 \leqq j \leqq r$, the k-derivation $\partial/\partial x_j$ can be extended to a K-derivation of $K[[y_1, \cdots, y_r]]$. Suppose that $\det(\partial f_i/\partial x_j) \notin q$. Then we can assume that $\det(\partial f_i/\partial x_j)_{1\leqq i\leqq r, 1\leqq j\leqq r}$ is not contained in $q(R_p)$ by Lemma 5. Then by Lemma 4,
$(f_1, \cdots, f_r)R_p = qR_p$.
There is an $s \in R\text{-}p$ such that
$sf_j \in (f_1, \cdots, f_r)$ for $r+1 \leqq j \leqq d$.
Then $s^{d-r}\det(\partial f_i/\partial x_j) \equiv 0 \bmod q$.
This implies $\det(\partial f_i/\partial x_j) \in q$, a contradiction.
Therefore $\det(\partial f_i/\partial x_j) \in q$ in this case.

Case 2. qR_p is not a complete intersection.

We choose generators $g_1, \cdots, g_n$ of $q(R_p)$ so that any r of $g_1, \cdots, g_n$ form a system of parameters of (R_p). Since $\det(\partial f_i/\partial x_j)$ is an (R_g)-linear combination of the elements of the form $\det(\partial h_i/\partial x_j)_{1\leqq i\leqq r, 1\leqq j\leqq r}$, where $h_1, \cdots, h_r$ is a system of parameters of (R_p) contained in $q(R_p)$.
By Lemma 2, we have $\det(\partial f_i/\partial x_j) \in q$.

Now we can prove Theorem 3.

Proof of Theorem 3.

Suppose that ht $(f_1, \cdots, f_d) < d$.
If $(f_1, \cdots, f_d) = q_1 \cap \cdots q_m \cap Q$ is a primary decomposition of $(f_1, \cdots, f_d)$, where Q is primary to m then det $(\partial f_i/\partial x_j) \in q_k$ $(1 \leqq k \leqq m)$ by Lemma 6 and, by Lemma 2 and Lemma 4 we have det $(\partial f_i/\partial x_j) \in Q$.

Corollary 6. Let $f \in m^2$. Then R/fR has an isolated singularity if and only if the Hessian det $(\partial^2 f/\partial x_i \partial x_j)$ is not contained in $(\partial f/\partial x_1, \cdots, \partial f/\partial x_d)$.

References

[K] Kunz, E., Kähler Differentials, Vieweg, Braunschweig/Wiesbaden, 1986.
[M] Matsumura, H., Commutative ring theory, Cambridge University Press, Cambridge, 1986.