Endomorphisms of Boolean Algebra

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Abstract

We study endomorphisms $\ell = x_1 + \cdots + x_n$ and $\partial = \partial_1 + \cdots + \partial_n$ of a boolean algebra A.
We express A as the direct sum of the eigen spaces of $\ell\partial$ or $\partial\ell$.
As a corollary, we give an elementary and direct proof of strong Lefscetz property of A.

Key words

Bollean algebra, eigenvalue, Lefshetz property

Let K be an algebraically closed field of characteristic 0 and let

$$A = K[X_1, \ldots, X_n]/(X_1^2, \ldots, X_n^2)$$

be a boolean algebra in n variables over K. We put $x_i = X_i \bmod (X_1^2, \ldots, X_n^2)$.

We define K-endomorphisms ℓ and ∂ of A as follows:

$$\ell(a) = (x_1 + \cdots + x_n)\, a \text{ for } a \in A, \text{ and}$$
$$\partial\ (x_{\alpha 1} \cdots x_{\alpha i}) = \sum_k x_{\alpha_1} \cdots \hat{x}_{\alpha_k} \cdots x_{\alpha_i} \text{ for a monomial } x_{\alpha_1} \cdots x_{\alpha_i} \text{ of } A.$$

We remark that ∂ is induced by a derivation $\sum_i \partial/\partial_{xi}$ of $K[X_1, \ldots, X_n]$, but is *not* a derivation of A.

Let A_i is the set of homogenous elememnts of degree i of A. Then

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_n$$

as a K-vector space. We put $\ell_i = \ell|_{A_i}$ and $\partial_i = \partial|_{A_i}$ for $0 \le i \le n$. Further we put $A_{-1} = A_{n+1} = (0)$, and define $l_{-1} = 0$ and $\partial_{n+1} = 0$ (zero maps).

LEMMA 1 . *For* $0 \le i \le n$, *we have*

$$\partial_{i+1}\ell_i - \ell_{i-1}\partial_i = (n - 2i)\, 1_{Ai}\,.$$

Proof. For a monomial $m_\alpha = \prod_j x^{\alpha_j}$ in A_i, we put $\alpha = \{\alpha_1, \ldots, \alpha_i\}$ (a subset of $n = \{1, \ldots, n\}$). We have

$$(\partial_{i+1}\ell_i - \ell_{i-1}\partial_i)(m_\alpha) = \partial_{i+1}\Big(\sum_{\beta \in n \setminus \alpha} x_\beta m_\alpha\Big) - \ell_{i-1}\Big(\sum_{j=1}^{i} \frac{m_\alpha}{x_{\alpha_j}}\Big)$$
$$= \sum_{\beta \in n \setminus \alpha} \left(m_\alpha + \sum_{j=1}^{i} x_\beta \frac{m_\alpha}{x_{\alpha j}} \right) - \sum_{j=1}^{i} \left(m_\alpha + \sum_{\beta \in n \setminus \alpha} x_\beta \frac{m_\alpha}{x_{\alpha_j}} \right)$$
$$= (n - i)\, m_\alpha - i m_\alpha = (n - 2i)\, m_\alpha. \qquad \square$$

Let M_i be the set of monomials of A of degree $i\,(0 \le i \le n)$. Then M_i is a K-basis of A_i.

We denote by M_i (resp. N_i) the matrix of ℓ_i (resp. ∂_i) with respect to M_i ($0 \le i \le n$).

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LEMMA 2. (1) $N_i = {}^tM_{i-1}$ for $1 \leq i \leq n$.

(2) $M_i = N_{n-i}$ *for* $0 \leq i \leq n-1$.

(3) $M_i = {}^tM_{n-i-1}$ for $0 \leq i \leq n-1$.

(4) $N_i = {}^tN_{n-i+1}$ for $1 \leq i \leq n$.

Proof. For the monomials $m_\alpha = x_{\alpha_1} \cdots x_{\alpha_i} \in M_i$ and $m_\beta = x_{\beta_1} \cdots x_{\beta_{i+1}} \in M_{i+1}$,

$$m_\alpha \text{ occurs in } \partial_i(m_\beta) \Leftrightarrow m_\beta \text{ occurs in } \ell_{i-1}(m_\alpha)$$

$$\Leftrightarrow \frac{x_1 \cdots x_n}{m_\beta} \text{ occurs in } \partial_{n-i}\left(\frac{x_1 \cdots x_n}{m_\alpha}\right).$$

Hence we have (1) and (2).

(3) and (4) follows from (1) and (2). □

COROLLARY 3. *The endmorphisms $\partial_{i+1}\ \ell_i$ and $\ell_{i-1}\ \partial_i$ are diagonalizable.*

Proof. By LEMMA 2, the matrix of $\partial_{i+1}\ell_i$(resp. $\ell_{i-1}\partial_i$) with respect to a basis M_i is $M_i{}^tM_i$ (resp. ${}^tM_{i-1} M_{i-1}$). Since these matrices are symmetric and their components are integres, $\partial_{i+1}\ell_i$ and $\ell_{i-1}\partial_i$ are diagonalizable. □

Let E_i be the set of eigenvalues of $\partial_{i+1}\ell_i$ and let E_i' be the set of eigenvalues of $\ell_{i-1}\partial_i$ for $0 \leq i \leq n$.

PROPSITION 4. *For $0 \leq i \leq [(n-1)/2]$, we have*

(1) $E_i = \{(n-2i+j)(j+1) \mid 0 \leq j \leq i\}$.

(1)' $E_i' = E_{i-1} \cup \{0\}$.

(2) $E_i = E'_{n-i}$ and $E_i' = E_{n-i}$.

Proof. Since $\dim A_{i-1} \leq \dim A_i$, (1)' follows from the following well known result:

SUBLEMMA 5. *Let M be an (r, s) matrix, N an (s, r) matrix in K. Then*

$$\Phi(MN, t) = t^{r-s}\, \Phi(NM, t),$$

where $\Phi(\ , t)$ denotes the charactristic polynomial of a matrix.

We prove (1) by induction on i.

The case $i = 0$ follows immediately from LEMMA 1.

Assume $i > 0$. We have $E_i = \{\lambda + n - 2i \mid \lambda \in E_i'\}$ by LEMMA 1, and $E_i' = \{(n-2(i-1)+j)(j+1) \mid 0 \leq j \leq i-1\} \cup \{0\}$ by (1)'. Hence we get the assertion by the induction hypothesis. □

(2) is clear by LEMMA 2.

COROLLARY 6. (1) *$\partial_{i+1}\ell_i$ is bijective for $0 \leq i \leq [(n-1)/2]$.*

(2) *$\ell_{i-1}\partial_i$ is bijective for $[n/2] + 1 \leq i \leq n$.*

Proof. (1) By (1) of PROPOSITION 4, each eigenvalue of $\partial_{i+1}\ell_i$ is a positive integer for $0 \leq i \leq [(n-1)/2]$, hence the assertion follows.

Similaraly (2) follows from (2) of PROPOSITION 4. □

COROLLARY 7. (1) *ℓ_i is injective for $0 \leq i \leq [(n-1)/2]$ and surjective for $[\mathrm{n}/2] \leq i \leq n$.*

(2) ∂_i is *surjective for $0 \leq i \leq [(n-1)/2]$ and injective for $[\mathrm{n}/2] \leq i \leq n$.*

Proof. Clear by COROLLARY 6. □

We put

$$A_i(\lambda) = \{a \in A_i \mid \partial_{i+1}\ell_i(a) = \lambda a\}, \text{ and}$$

$$A_i'(\lambda) = \{a \in A_i \mid \ell_{i-1}\,\partial_i(a) = \lambda a\}\,(0 \leq i \leq n)$$

for $\lambda \in K$

LEMMA 8. *For* $0 \leq i \leq n$,

(1) $A_i(\lambda) = A_i'(\lambda - n + 2i)$.

(2) $A_i(0) = \mathrm{Ker}\ \ell_i$.

(3) $A_i'(0) = \mathrm{Ker}\ \partial_i$.

Proof. (1) Immediate from LEMMA 1.

(2) For $0 \leq i < [n/2]$, $A_i(0) = \mathrm{Ker}\ \ell_i = (0)$ by COROLLARY 6 and 7. For $i \geq [n/2]$, ∂_{i+1} is injective and hence $\mathrm{Ker}\ \ell_i = \mathrm{Ker}\partial_{i+1}\ell_i = A_i(0)$.

The proof of (3) is similar to (2) and we omit it. □

PROPOSITION 9. *For* $0 \leq i \leq n$ and $\lambda \in E_i \setminus \{0\}$, *we have*

$$\ell_i|A_i(\lambda): A_i(\lambda) \rightrightarrows A_{i+1}(\lambda + n - 2(i+1)).$$

Proof. By LEMMA 8, it suffices to show $\ell_i|_{A_i}(\lambda): A_i(\lambda) \rightrightarrows A'_{i+1}(\lambda)$. Assume $a \in A_i(\lambda)$. Then $\ell_i\partial_{i+1}\ell_i(a) = \ell_i(\lambda a)) = \lambda\,\ell_i(a))$, hence $\ell_i(a) \in A'_{i+1}(\lambda)$.

Since $\lambda \neq 0$, $\ell_i|A_{i(\lambda)}$ is injective. Moreover if b is an element of $A'_{i+1}(\lambda)$, we have $b = \ell_i\partial_{i+1}(\lambda^{-1} b)$. Hence $\ell_i|_{A_i(\lambda)}: A_i(\lambda) \rightrightarrows A'_{i+1}(\lambda)$ is surjective. □

Now we can prove the main result.

THEOREM 10. (1) $A_i = \ell_{i-1}(A_{i-1}) \oplus \mathrm{Ker}\ \partial_i$ *for* $0 \leq i \leq [(n+1)/2]$.

(2) $A_i \simeq \bigoplus_{\lambda \neq 0} A_{i-1}$ for $[n/2] + 1 \leq i \leq n$.

Proof. (1) By LEMMA 8 and PROPOSITION 9,

$$\begin{aligned} A_i &= \bigoplus_{\lambda \in E_i} A_i(\lambda) \\ &= (\bigoplus_{n-2i \neq \lambda \in E_i} A_i(\lambda)) \oplus A'_i(0) \\ &= (\bigoplus_{\lambda \in E_{i-1}} \ell_{i-1}(A_{i-1}(\lambda))) \oplus \mathrm{Ker}\ \partial_i \\ &= \ell_{i-1}(A_{i-1}) \oplus \mathrm{Ker}\ \partial_i. \end{aligned}$$

for $0 \leq i \leq [(n+1)/2]$.

(2) If $[n/2] + 1 \leq i \leq n$, then

$$\begin{aligned} A_i &= \ell_i(A_{i-1}) \\ &= \ell_i(\bigoplus_{\lambda \in E_{i-1}} A_{i-1}(\lambda)) \\ &= \ell_i(\bigoplus_{0 \neq \lambda \in E_{i-1}} A_{i-1}(\lambda)) \\ &\simeq \bigoplus_{0 \neq \lambda \in E_{i-1}} A_{i-1}(\lambda) \end{aligned}$$

□

COROLLARY 11. *For* $0 \leq i \leq [n/2]$

(1) *The K-endomorphism* $\ell_{n-i-1} \cdots \ell_i : A_i \to A_{n-i}$ *is an isomorphism.*

(2) *The K-endomorphism* $\partial_{i+1} \cdots \partial_{n-i} : A_{n-i} \to A_i$ *is an isomorphisim.*

REMARK 12. (1) of COROLLARY 11 means that A has the strong Lefshetz property. Although this is already known (see e.g. [1]), here we gave an elementary proof.

References

[1] J. Watanabe, The Dilworth number of Artinian rings and finite posets with rank function, j Adv. Stud, Pure Math., 11 (1987), 303–312

[2] H. Morita, A. Wachi and J. Watanbe, Zero-dimensional Gorenstein algebras with the action of the symmetric group S_k, in preprint